

EFFECTIVE RESOLUTION OF DIOPHANTINE EQUATIONS OF THE FORM $u_n + u_m = wp_1^{z_1} \cdots p_s^{z_s}$

ISTVÁN PINK AND VOLKER ZIEGLER

ABSTRACT. Let u_n be a fixed non-degenerate binary recurrence sequence with positive discriminant, w a fixed non-zero integer and p_1, p_2, \dots, p_s fixed, distinct prime numbers. In this paper we consider the Diophantine equation $u_n + u_m = wp_1^{z_1} \cdots p_s^{z_s}$ and prove under mild technical restrictions effective finiteness results. In particular we give explicit upper bounds for n, m and z_1, \dots, z_s . Furthermore, we provide a rather efficient algorithm to solve Diophantine equations of the described type and we demonstrate our method by an example.

1. INTRODUCTION

Recently several authors considered the problem to find all indices n and m and exponents z such that

$$u_n + u_m = 2^z,$$

where u_n is a fixed recurrence. In particular Bravo and Luca considered the cases, where u_n is the Fibonacci sequence [4] and the Lucas sequence [5] respectively. Also the case where u_n is the generalized k -Fibonacci sequence has been considered independently by Bravo, Gómez and Luca [3] and Marques [9]. In [2] Bertók, Hajdu, Pink and Rábai solved completely equations of the form $u_n = 2^a + 3^b + 5^c$, where u_n is one of the Fibonacci, Lucas, Pell and associated Pell sequences, respectively.

In the present paper we aim to generalize the results due to Bravo and Luca [5, 4] and consider the more general Diophantine equation

$$(1) \quad u_n + u_m = wp_1^{z_1} \cdots p_s^{z_s}$$

in non-negative integer unknowns n, m, z_1, \dots, z_s , where $\{u_n\}_{n \geq 0}$ is a binary non-degenerate recurrence sequence, p_1, \dots, p_s are distinct primes and w is a non-zero integer with $p_i \nmid w$ for $1 \leq i \leq s$. For reasons of symmetry we assume that $n \geq m$. Although the ideas of Bravo, Gómez, Luca and Marques [5, 4, 3, 9] can be easily extended to this case, we provide two new aspects treating the title Diophantine equation.

Firstly, the main argument due to Bravo et.al. is to consider two linear forms in (complex) logarithms and obtain by a clever trick upper bounds for n, m and z . In our approach we replace one of the two linear forms in complex logarithms by linear forms in p_i -adic logarithms for every prime p_i with $1 \leq i \leq s$. The advantage in doing this is that instead of considering a linear form in $s+2$ logarithms we only have to consider a linear form in two p -adic logarithms which reduces the upper bound for n, m, z_1, \dots, z_s drastically for large s .

2010 *Mathematics Subject Classification.* 11D61, 11B39, 11Y50.

Key words and phrases. Lucas sequences, S-units, automatic resolution.

The research was granted by the Austrian science found (FWF) under the project P 24801-N26.

The second novelty in treating this kind of problems is a consequence to our p -adic approach by using a p -adic reduction method due to Pethő and de Weger [12] instead of real (and complex) approximation lattices or the method of Baker and Davenport [1]. This approach leads to a better performance of the reduction step and a rather efficient algorithm.

Before we state our main result, let us fix some notations. We call the sequence $\{u_n\}_{n \geq 0} = \{u_n(A, B, u_0, u_1)\}_{n \geq 0}$ a binary linear recurrence sequence defined over the integers if the relation

$$(2) \quad u_n = Au_{n-1} + Bu_{n-2} \quad (n \geq 2)$$

holds, where $A, B \in \mathbb{Z}$ with $AB \neq 0$ and u_0, u_1 are fixed rational integers with $|u_0| + |u_1| > 0$. The polynomial $f(x) = x^2 - Ax - B$ attached to recurrence (2) is the so-called companion polynomial of the sequence $\{u_n\}_{n \geq 0}$ and we denote by $\Delta = A^2 + 4B$ the discriminant of f . Let α and β be the roots of the companion polynomial f and assume that $\Delta \neq 0$, then it is well known that there exist constants $a = u_1 - u_0\beta$ and $b = u_1 - u_0\alpha$ such that the following formula holds

$$(3) \quad u_n = \frac{a\alpha^n - b\beta^n}{\alpha - \beta}.$$

The sequence $\{u_n\}_{n \geq 0}$ is called *non-degenerate*, if $ab\alpha\beta \neq 0$ and α/β is not a root of unity.

Throughout the paper we will assume that u_n is non-degenerate and that $\Delta > 0$. The last assumption implies that the sequence $\{u_n\}_{n \geq 0}$ possesses a dominant root, which means that without loss of generality $|\alpha| > |\beta|$. Under these assumptions and notations the main result of our paper is now as follows:

Theorem 1. *Let $\{u_n\}_{n \geq 0}$ be a non-degenerate binary recurrence with $\Delta > 0$. Let us assume that $p_i \nmid \gcd(A, B)$ for all $1 \leq i \leq s$ and furthermore let us assume that none of the following two conditions hold*

- $\beta = \pm 1$ and $m = \frac{\log(\beta^m 2b/a)}{\log \alpha}$.
- $\beta = -1$ and there exists an positive odd integer x and integers t_1, \dots, t_s such that

$$(4) \quad \frac{w(\alpha + 1)}{a(\alpha^x + 1)} = p_1^{-t_1} \dots p_s^{-t_s}.$$

Then there exists an effectively computable constant C depending only on $\{u_n\}_{n \geq 0}$, w, s, p_1, \dots, p_s such that all solutions (n, m, z_1, \dots, z_s) to equation (1) satisfy

$$\max\{n, m, z_1, \dots, z_s\} < C.$$

Let us stress out that during the course of proof of Theorem 1 we give a very precise description of how to compute this bound C (see for instance Proposition 3 and Section 10). Moreover, we present an easy implementable algorithm to solve Diophantine equation (1) completely (see Section 8) under the assumptions of Theorem 1.

Let us discuss the technical assumptions made in Theorem 1. First, the assumption that $p_i \nmid \gcd(A, B)$ for all $1 \leq i \leq s$ is to avoid technical difficulties. Using instead of lower bounds for linear forms of p -adic logarithms lower bounds for linear forms of complex logarithms in Section 3 one can avoid these difficulties, e.g. by adopting the method due to Bravo and Luca [4]. However we do not intend to further discuss this case in detail.

In the case that the first exceptional case of Theorem 1 holds, i.e. that $\beta = \pm 1$ and $m = \frac{\log(\beta^m 2b/a)}{\log \alpha}$, equation (1) can be rewritten as

$$\frac{a\alpha^n}{\alpha \pm 1} = wp_1^{z_1} \dots p_s^{z_s}.$$

It is easy to see that this Diophantine equation may have infinitely many solutions which are easy to determine. In particular, it is possible to determine all solutions to such an equation by slightly adapting our method. Note that α has to be a rational integer if $\beta = \pm 1$. For instance, let us choose $a = b = \beta = w = s = 1$ and $\alpha = p_1 = 2$. Then equation (1) turns into

$$\frac{2^n - 1}{2 - 1} + \frac{2^m - 1}{2 - 1} = 2^{z_1}$$

which has infinitely many solutions of the form $m = 1$ and $n = z_1$.

Let us turn to the second exceptional case of Theorem 1. If $\beta = -1$ and $n - m$ is odd and let us write $x = n - m > 0$, then Diophantine equation (1) turns into

$$\frac{a\alpha^n + b(-1)^n}{\alpha + 1} + \frac{a\alpha^m + b(-1)^m}{\alpha + 1} = \frac{a\alpha^m(\alpha^x + 1)}{\alpha + 1} = wp_1^{z_1} \dots p_s^{z_s}.$$

If we insert now the assumed relation (4) we get

$$\alpha^m = p_1^{z_1 - t_1} \dots p_s^{z_s - t_s}$$

and it is easy to see that this Diophantine equation may have infinitely many solutions. Moreover, also in this case it is easy to determine all solutions to such an equation. In view of Theorem 1 only the question how to find all solutions to equation (4) remains to solve Diophantine equation (1) completely in any case. However, Diophantine equation (4) is nothing else than

$$u_x = w' p_1^{t_1} \dots p_s^{t_s}$$

with $w' = w/a$. This type of Diophantine equation has been studied by Pethő and de Weger [12] and they gave a practical algorithm how to solve such equations. Moreover, we will also discuss this kind of Diophantine equation in Section 4. Let us provide an example with infinitely many solutions in this case. For instance let $a = b = w = 1$, $\beta = -1$ and $\alpha = p_1 \dots p_s$. Then it is easy to see that equation (1) has infinitely many solutions of the form $n = m + 1$ and $z_1 = \dots = z_s = m$.

Finally let us outline the plan of the paper. The next three sections will provide a proof of Theorem 1. First, we introduce some further notations and prove some auxiliary results in Section 2. In Section 3 we use lower bounds of linear forms in two p -adic logarithms due to Bugeaud and Laurent [6] in order to prove some kind of gap principle. In the Sections 4 and 5 we exploit ideas due to Pethő and de Weger [12] and Bravo and Luca [5, 4] respectively. This leads us to our upper bound C in Theorem 1. Due to the use of Baker's method the constant C in Theorem 1 is usually very large. By using the LLL-algorithm and ideas due to de Weger [7] and Pethő and de Weger [12] it is possible to reduce these bounds considerably in concrete examples. In Section 7 we will discuss the method of de Weger [7] and Pethő and de Weger [12] and show how to apply them to our problem. All together this provides an algorithm to solve Diophantine equations of type (1) completely (see Section 8). In order to demonstrate the efficiency of our algorithm we solve the Diophantine equation

$$u_n + u_m = 2^{z_1} 3^{z_2} \dots 199^{z_{46}}$$

completely, where u_n is either the Fibonacci sequence or the Lucas sequence (see Section 9). Since it is hard to track all the constants which appear in the proof of Theorem 1, we provide in the final section a list of all constants and their explicit determination.

2. NOTATIONS AND AUXILIARY RESULTS

In this section we keep the notations of the introductory section. However, before we start with the proof of Theorem 1 we need to introduce some more notations. For a positive real number $x > 0$ we define the function \log_* by

$$\log_* : \mathbb{R}_{>0} \rightarrow \mathbb{R} \quad \log_* x := \max\{0, \log x\}.$$

By $\varphi = \frac{1+\sqrt{5}}{2}$ we denote the golden ratio. Finally we write $K = \mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\sqrt{\Delta})$ for the number field corresponding to our binary sequence $\{u_n\}_{n \geq 0}$ and define $d_K = [K : \mathbb{Q}]$.

In the rest of the paper there will appear constants c_1, c_2, \dots and also occasionally constants of slightly different form, e.g. $c_{8,i}$, which are all explicitly computable. Sometimes we do not state them explicitly in our results or proofs for aesthetic reasons. However, Section 10 provides a list of all constants and their explicit determination. We advise readers who wish to keep track of all constants and their explicit determination to keep a bookmark at Section 10.

The first lemma is an elementary result due to Pethő and de Weger [12]. It will be used over and over again in the proof of Theorem 1. For a proof of Lemma 1 we refer to [14, Appendix B].

Lemma 1. *Let $u, v \geq 0, h \geq 1$ and $x \in \mathbb{R}$ be the largest solution of $x = u + v(\log x)^h$. Then*

$$x < \max\{2^h(u^{1/h} + v^{1/h} \log(h^h v))^h, 2^h(u^{1/h} + 2e^2)^h\}.$$

Let us assume from now on that (n, m, z_1, \dots, z_s) is a solution to Diophantine equation (1) satisfying the assumptions made in Theorem 1. The next lemma gives lower and upper bounds for the quantity $|u_n + u_m|$, as well as, upper and lower bounds for the exponents z_i in terms of n . These bounds will be utilized in the proof of Theorem 1.

Lemma 2. *There exist constants c_1, \dots, c_5 such that the following holds:*

- (i) $|u_n + u_m| < c_1 |\alpha|^n$.
- (ii) *If $n > c_2$, then we have $z_i < \frac{2 \log |\alpha|}{\log p_i} n$ for $i = 1, \dots, s$. In particular, for all $1 \leq i \leq s$ we have $z_i < \frac{2 \log |\alpha|}{\log 2} n$.*
- (iii) *Provided that $n > c_3$, we have*

$$c_4 |\alpha|^n < |u_n + u_m|.$$

- (iv) *Provided that $n > c_3$, we have*

$$n < \frac{\sum_{i=1}^s z_i \log p_i}{\log |\alpha|} + c_5.$$

Proof. Let us start with (i). We note that since $|\alpha| > |\beta|$ and $|\alpha - \beta| = \sqrt{\Delta}$, we get by the triangle inequality

$$|u_n| = \left| \frac{a\alpha^n - b\beta^n}{\alpha - \beta} \right| < \frac{|a| + |b|}{\sqrt{\Delta}} |\alpha|^n.$$

Thus, the above inequality leads to

$$|u_n + u_m| \leq |u_n| + |u_m| < \frac{|a| + |b|}{\sqrt{\Delta}} |\alpha|^n + \frac{|a| + |b|}{\sqrt{\Delta}} |\alpha|^m,$$

which by $n \geq m$ gives $|u_n + u_m| < c_1 |\alpha|^n$, with $c_1 = 2 \frac{|a| + |b|}{\sqrt{\Delta}}$.

Next, we prove (ii). By combining equation (1) and the just proved part (i) of the lemma we may write for every $1 \leq i \leq s$

$$|w|p_i^{z_i} \leq |w|p_1^{z_1} \cdots p_s^{z_s} = |u_n + u_m| < c_1 |\alpha|^n.$$

Thus, by taking logarithms the above inequality implies

$$z_i \log p_i \leq n \log |\alpha| \left(1 + \frac{c_2}{n}\right),$$

with $c_2 = \frac{\log_* \frac{c_1}{|w|}}{\log |\alpha|}$. Assuming that $n > c_2$ we get

$$(5) \quad z_i < \frac{2n \log |\alpha|}{\log p_i}$$

and the first inequality of (ii) follows. The second inequality of (ii) is a simple consequence of (5) by noting that $p_i \geq 2$ for every $1 \leq i \leq s$.

For proving (iii) we note that since α and β are roots of the quadratic monic polynomial $x^2 - Ax - B$ with $A, B \in \mathbb{Z}$ and $\Delta = A^2 + 4B > 0$ we have

$$|\alpha| \geq \left| \frac{-A \pm \sqrt{A^2 + 4B}}{2} \right| \geq \frac{|A| + \sqrt{|A|^2 + 4|B|}}{2} \geq \frac{1 + \sqrt{5}}{2} = \varphi.$$

Since we assume that $n \geq m$ we find

$$\left| 1 + \frac{1}{\alpha^{n-m}} \right| \geq 1 - \frac{1}{\varphi},$$

hence we obtain by formula (3)

$$(6) \quad \begin{aligned} |u_n + u_m| &= \frac{1}{\sqrt{\Delta}} \left| a\alpha^n \left(1 + \frac{1}{\alpha^{n-m}} \right) - (b\beta^n + b\beta^m) \right| \geq \\ &\geq \frac{1}{\sqrt{\Delta}} \left| |a||\alpha|^n \left(1 - \frac{1}{\varphi} \right) - |b|(|\beta|^n + |\beta|^m) \right|. \end{aligned}$$

Let us distinguish between the case that $|\beta| \leq 1$ and $|\beta| > 1$.

In the case that $|\beta| \leq 1$ inequality (6) yields

$$\begin{aligned} |u_n + u_m| &\geq \frac{1}{\sqrt{\Delta}} \left| |a||\alpha|^n \left(1 - \frac{1}{\varphi} \right) - 2|b| \right| = \\ &= \frac{1}{\sqrt{\Delta}} |a||\alpha|^n \left(1 - \frac{1}{\varphi} \right) \left| 1 - \frac{2|b|}{|a||\alpha|^n \left(1 - \frac{1}{\varphi} \right)} \right|. \end{aligned}$$

Let us assume that n is large enough such that $1 - \frac{2|b|}{|a||\alpha|^n \left(1 - \frac{1}{\varphi} \right)} > \frac{1}{2}$, i.e. $n > c' =$

$\frac{\log_* \frac{4|b|\varphi}{|a|(\varphi-1)}}{\log |\alpha|}$. Thus, we obtain

$$|u_n + u_m| > \frac{1}{2\sqrt{\Delta}} |a||\alpha|^n \left(1 - \frac{1}{\varphi} \right) = c_4 |\alpha|^n,$$

with $c_4 = \frac{|a|(\varphi-1)}{2\varphi\sqrt{\Delta}}$.

Now let us assume that $|\beta| > 1$, then by $n \geq m$ and inequality (6) we get

$$\begin{aligned} |u_n + u_m| &\geq \frac{1}{\sqrt{\Delta}} \left| |a||\alpha|^n \left(1 - \frac{1}{\varphi}\right) - 2|b||\beta|^n \right| = \\ &= \frac{1}{\sqrt{\Delta}} |a||\alpha|^n \left(1 - \frac{1}{\varphi}\right) \left| 1 - \frac{2|b||\beta|^n}{|a||\alpha|^n \left(1 - \frac{1}{\varphi}\right)} \right|. \end{aligned}$$

If we assume that n is large enough, say $n > c'' = \frac{\log_* \frac{4|b|\varphi}{|a|(\varphi-1)}}{\log \frac{|\alpha|}{|\beta|}}$, then we get that

$$\left| 1 - \frac{2|b||\beta|^n}{|a||\alpha|^n \left(1 - \frac{1}{\varphi}\right)} \right| > \frac{1}{2}. \text{ Thus also in this case we have}$$

$$|u_n + u_m| > \frac{1}{2\sqrt{\Delta}} |a||\alpha|^n \left(1 - \frac{1}{\varphi}\right) = c_4 |\alpha|^n,$$

which proves (iii) in the case that $|\beta| > 1$. All together we have proved (iii) assuming that $n > c_3 = \max\{c', c''\}$.

Finally we turn to the proof of (iv). Let us note that by assuming that n is large enough, i.e. $n > c_3$, we may combine the inequality from (iii) with equation (1). Thus we get

$$c_4 |\alpha|^n < |u_n + u_m| = |w| p_1^{z_1} \dots p_s^{z_s},$$

and by taking logarithms this yields

$$n \log |\alpha| < \log \frac{|w|}{c_4} + \sum_{i=1}^s z_i \log p_i$$

which proves (iv), with $c_5 = \frac{\log \frac{|w|}{c_4}}{\log |\alpha|}$. \square

Before stating the next result let us introduce some further notations. Let L be a number field and $\eta \in L$. We denote by $h(\eta)$ as usual the *absolute logarithmic Weil height* of η , i.e.

$$h(\eta) = \frac{1}{d} \left(\log |a_0| + \sum_{i=1}^d \log_* (|\eta^{(i)}|) \right),$$

where a_0 is the leading coefficient of the minimal polynomial of η over \mathbb{Z} and the $\eta^{(i)}$ -s are the conjugates of η in \mathbb{C} . We will use the following well known properties of the absolute logarithmic height without special reference:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta \gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^\ell) &= |\ell| h(\eta), \quad \text{for } \ell \in \mathbb{Z}. \end{aligned}$$

In general it is a very hard problem to find lower bounds for the height of elements in a number field of given degree. However, in the case of quadratic fields the problem can be solved easily:

Lemma 3. *For an algebraic number α of degree two we have $h(\alpha) \geq 0.24$ or α is a root of unity.*

Proof. Assume that the conjugate of α is β and that $|\alpha| \geq |\beta|$. If α is not an algebraic integer, then the leading coefficient a_0 of its minimal polynomial satisfies $a_0 \geq 2$ and we obtain

$$h(\alpha) \geq \frac{1}{2} \log |a_0| \geq \frac{\log 2}{2} > 0.34.$$

On the other hand if α is an algebraic integer, then $|\alpha| \geq \varphi$ and

$$h(\alpha) \geq \frac{1}{2}(\log_* |\alpha| + \log_* |\beta|) \geq \frac{\log \varphi}{2} > 0.24.$$

□

3. THE APPLICATION OF p -ADIC METHODS

Our next goal is to prove an upper bound for n in terms of $n - m$ and $\log n$. Therefore we utilize linear forms in p -adic logarithms. In particular, we use the results due to Bugeaud and Laurent [6]. This result has the drawback that we have to assume that the algebraic numbers involved in the p -adic logarithms are linearly independent in contrast to the results due to Yu [16, 17] which can be applied without this assumption. Thus dealing with the linearly dependent case causes some technical difficulties. Nevertheless we use the result due to Bugeaud and Laurent [6] since their result yields rather small upper bounds and dealing with the technical difficulties will pay off when we come to the LLL-reduction step.

For a prime number p denote by \mathbb{Q}_p the field of p -adic numbers with the standard p -adic valuation ord_p . Further, let α_1, α_2 be algebraic numbers over \mathbb{Q} and we regard them as elements of the field $L = \mathbb{Q}_p(\alpha_1, \alpha_2)$. We equip the field L with the ultrametric absolute value $|x|_p = p^{-\nu_p(x)}$, where ν_p denotes the unique extension to L of the standard p -adic valuation ord_p over \mathbb{Q}_p normalized by $\nu_p(p) = 1$ (we set $\nu_p(0) = +\infty$). Note, that for every non-zero $\delta \in L$ we have

$$\nu_p(\delta) = \frac{\text{ord}_p(N_{L/\mathbb{Q}_p}(\delta))}{d_L},$$

where $d_L = [L : \mathbb{Q}_p]$ is the degree of the field extension L/\mathbb{Q}_p and $N_{L/\mathbb{Q}_p}(\delta)$ is the norm of δ with respect to \mathbb{Q}_p . Denote by e the ramification index of the local field extension L/\mathbb{Q}_p and by f the residual degree of this extension. Put

$$\mathcal{D} = \frac{[\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]}{f}$$

and let

$$h'(\alpha_i) \geq \max \left\{ h(\alpha_i), \frac{\log p}{\mathcal{D}} \right\}, \quad (i = 1, 2).$$

Before stating the above mentioned result of Bugeaud and Laurent we introduce p -adic logarithms (see e.g. [15, Chapter 5.1] for a short but thorough treatment of this topic). Let \mathcal{K} be a field complete with respect to $|\cdot|_p$ and containing \mathbb{Q}_p . There exists a function $\log_p(x)$ defined on all of \mathcal{K} such that $\log_p(xy) = \log_p(x) + \log_p(y)$. Moreover, for every $\xi \in \mathcal{K}$ with $|\xi - 1|_p < p^{-1/(p-1)}$ we can compute the p -adic logarithm by

$$\log_p(\xi) = - \sum_{i=1}^{\infty} \frac{(1 - \xi)^i}{i}.$$

Moreover we have

$$(7) \quad |\log_p(\xi)|_p = |\xi - 1|_p$$

or equivalently

$$(8) \quad \nu_p(\log_p(\xi)) = \nu_p(\xi - 1)$$

provided that $|\xi - 1|_p < p^{-1/(p-1)}$. Note that in the case that $|\xi - 1|_p = p^{-1/(p-1)}$ we can replace in equation (7) the $=$ by a \leq sign, (or equivalently in equation (8) the $=$ by a \geq sign).

With these notations at hand, let us state a result due to Bugeaud and Laurent [6, Corollary 1]):

Theorem 2. *Let b_1, b_2 be positive integers and suppose that α_1 and α_2 are multiplicatively independent algebraic numbers such that $\nu_p(\alpha_1) = \nu_p(\alpha_2) = 0$. Put*

$$(9) \quad b' := \frac{b_1}{\mathcal{D}h'(\alpha_2)} + \frac{b_2}{\mathcal{D}h'(\alpha_1)}.$$

Then we have

$$(10) \quad \nu_p(\alpha_1^{b_1} \alpha_2^{b_2} - 1) \leq \frac{24p(p^f - 1)\mathcal{D}^4}{(p - 1)(\log p)^4} B^2 h'(\alpha_1) h'(\alpha_2),$$

with

$$(11) \quad B := \max \left\{ \log b' + \log \log p + 0.4, 10, \frac{10 \log p}{\mathcal{D}} \right\}.$$

We only use this result in the case where $L = \mathbb{Q}_p(\sqrt{\Delta})$, i.e. $1 \leq d_L, \mathcal{D}, e, f \leq 2$ and obtain as a simple Corollary

Corollary 1. *Let $B' = \max\{b_1, b_2, p^{10}, e^{10}\}$ and with the notations and assumptions of Theorem 2 in force we have*

$$\nu_p(\alpha_1^{b_1} \alpha_2^{b_2} - 1) < C_1(p) h'(\alpha_1) h'(\alpha_2) (\log B')^2,$$

where

$$C_1(p) = \frac{947p^f}{(\log p)^4}.$$

Proof. First let us note that since $f, \mathcal{D} \leq 2$ one easily deduces that

$$\frac{24p(p^f - 1)\mathcal{D}^4}{(p - 1)(\log p)^4} < \frac{768p^f}{\log^4 p}.$$

Moreover,

$$b' = \frac{b_1}{\mathcal{D}h'(\alpha_2)} + \frac{b_2}{\mathcal{D}h'(\alpha_1)} \leq \frac{b_1 + b_2}{\log p},$$

and since we assume that $B' \geq \max\{p^{10}, e^{10}\}$ we get

$$B = \max \left\{ \log b' + \log \log p + 0.4, 10, \frac{10 \log p}{\mathcal{D}} \right\} \leq \log(2e^{0.4} B') \leq 1.11 \log B'.$$

Putting everything together yields the Corollary. \square

Note that it is rather easy to compute e and f . Indeed, let Δ_0 be the square-free part of Δ , then we have $e = 2$ if and only if $p \mid \Delta_0$ and $f = 2$ if and only if $\left(\frac{\Delta_0}{p}\right) = -1$, where $\left(\frac{\Delta_0}{p}\right)$ denotes the Legendre symbol.

Proposition 1. *Let us assume that $n > m > 3$, $P = \max_{1 \leq i \leq s} \{p_i\}$ and that*

$$n > c_6 := \max \{c_3, 17.5 \log |\alpha| (\max \{\log |2a\alpha|, \log |2b\beta|\} + 0.24), P^{10}, e^{10}\}.$$

Then there exist constants c_7 and $c_{8,i}$ with $i = 1, \dots, s$ such that

$$n < c_7(n-m)(\log n)^2 \quad \text{and} \quad z_i < c_{8,i}(n-m)(\log n)^2,$$

with $1 \leq i \leq s$.

Proof. First, let us note that the first inequality is a direct consequence of the second inequality and Lemma 2 (iv) and c_7 can be easily computed once we have found the constants $c_{8,i}$ (see Section 10). Therefore we are left to prove the second inequality for each index $1 \leq i \leq s$. Let us fix the index i . In order to avoid an overload of indices we drop the index i for the rest of the proof, i.e. we write $p = p_i$, $c_8 = c_{8,i}$ and so on. Further, we will work in the field $L = \mathbb{Q}_p(\alpha, \beta) = \mathbb{Q}_p(\sqrt{\Delta})$. Since we assume $p \nmid \gcd(A, B)$ we have that $\nu_p(\alpha) = 0$ or $\nu_p(\beta) = 0$. Without loss of generality we may assume that $\nu_p(\alpha) = 0$, since we make no use of the fact that $|\alpha| > |\beta|$ in the whole proof of the proposition.

Let us consider equation (1) and rewrite it as follows

$$(12) \quad \frac{b(\beta^{n-m} + 1)}{a(\alpha^{n-m} + 1)} \left(\frac{\beta}{\alpha} \right)^m - 1 = -w(\alpha - \beta)\alpha^{-m}a^{-1}(\alpha^{n-m} + 1)^{-1}p_1^{z_1} \cdots p_s^{z_s}$$

Let us denote by Λ the left hand side of (12) and let us compute $\nu_p(\Lambda)$ by considering the right hand side of equation (12). Since by assumption $\nu_p(\alpha) = \nu_p(w) = 0$ we obtain

$$\nu_p(\Lambda) = z - \nu_p(a(\alpha^{n-m} + 1)) + \nu_p(\alpha - \beta).$$

Let us note that $\alpha - \beta$ is an algebraic integer, hence $\nu_p(\alpha - \beta) \geq 0$. Moreover, we have

$$\begin{aligned} \nu_p(a(\alpha^{n-m} + 1)) &\leq \frac{\log |a| + \log 2 + (n-m) \log |\alpha|}{\log p} \\ &\leq (n-m) \frac{\log |2a\alpha|}{\log p} := c_9(n-m). \end{aligned}$$

and obtain

$$(13) \quad \nu_p(\Lambda) \geq z - c_9(n-m).$$

Next, we will apply Corollary 1 in order to bound $\nu_p(\Lambda)$ from above. Therefore let us estimate the height of $\frac{b(\beta^{n-m} + 1)}{a(\alpha^{n-m} + 1)}$. Note that α and β (resp. a and b) are either rational integers or conjugate algebraic integers in a real quadratic field. Therefore we conclude that $h\left(\frac{\alpha}{\beta}\right) \leq \max\{\log |\alpha|, \log |\beta|\}$ provided that α and β are either rational integers or conjugate algebraic integers in a real quadratic field. Indeed this is clear in the rational case. In the real quadratic case under the assumption that $|\alpha| \geq |\beta|$ we get

$$\begin{aligned} h\left(\frac{\alpha}{\beta}\right) &= \frac{1}{2} \left(\log |a_0| + \log_* \left| \frac{\alpha}{\beta} \right| + \log_* \left| \frac{\beta}{\alpha} \right| \right) \\ &\leq \frac{1}{2} (\log |\alpha| + \log |\beta| + \log |\alpha| - \log |\beta|) \\ &= \log |\alpha|. \end{aligned}$$

Note that the leading coefficient a_0 of the minimal polynomial of α/β satisfies $a_0|\alpha\beta = B$. Therefore we obtain

$$\begin{aligned}
 (14) \quad h\left(\frac{b(\beta^{n-m}+1)}{a(\alpha^{n-m}+1)}\right) &\leq \max\{\log|a| + \log|\alpha^{n-m}+1|, \log|b| + \log|\beta^{n-m}+1|\} \\
 &\leq \max\{\log|a| + (n-m)\log|\alpha| + \log 2, \\
 &\quad \log|b| + (n-m)\log|\beta| + \log 2\} \\
 &\leq (n-m)\max\{\log|2a\alpha|, \log|2b\beta|\},
 \end{aligned}$$

hence

$$h'\left(\frac{b(\beta^{n-m}+1)}{a(\alpha^{n-m}+1)}\right) \leq (n-m)\max\{\log|2a\alpha|, \log|2b\beta|, \log p\} := c_{10}(n-m).$$

Furthermore note that $h'(\alpha/\beta) \leq \max\{\log|\alpha|, \log p\}$.

We have to distinguish now between two cases, namely whether $\frac{b(\beta^{n-m}+1)}{a(\alpha^{n-m}+1)}$ and α/β are multiplicatively independent or not. Let us deal with the dependent case first. In this case there exist co-prime integers r and s such that

$$\frac{a(\alpha^{n-m}+1)}{b(\beta^{n-m}+1)} = \left(\frac{\alpha}{\beta}\right)^{r/s}.$$

Let us find upper bounds for $|r|$ and $|s|$.

First, we note that $\delta := (\alpha/\beta)^{1/s}$ has to be still of degree two and is not a root of unity. Indeed $(\alpha/\beta)^{r/s} \in K$ therefore $(\alpha/\beta)^r \in K^s$. Since by assumption r and s are coprime we obtain that $(\alpha/\beta) \in K^s$ hence $\delta := (\alpha/\beta)^{1/s} \in K$. Hence

$$0.24 < \frac{h(\alpha/\beta)}{|s|} \leq \frac{1}{|s|} \log|\alpha|$$

and $|s| < 4.2 \log|\alpha|$ by Lemma 3. On the other hand we deduce from the upper bound (14) that

$$\begin{aligned}
 4.2(n-m)\max\{\log|2a\alpha|, \log|2b\beta|\} \log|\alpha| &\geq |s|h\left(\frac{b(\beta^{n-m}+1)}{a(\alpha^{n-m}+1)}\right) \\
 &= |r| \cdot h\left(\frac{\alpha}{\beta}\right) \geq |r|0.24
 \end{aligned}$$

i.e.

$$|r| < 17.5(n-m)\log|\alpha|\max\{\log|2a\alpha|, \log|2b\beta|\}.$$

In particular, we have

$$-ms - r \leq m|s| + |r| \leq 17.5n \log|\alpha| (\max\{\log|2a\alpha|, \log|2b\beta|\} + 0.24)$$

Therefore rewriting (12) we obtain

$$\begin{aligned}
 \nu_p(\delta^{-ms-r} - 1) &= \nu_p(-w(\alpha - \beta)\alpha^{-m}a^{-1}(\alpha^{n-m}+1)^{-1}p_1^{z_1} \dots p_s^{z_s}) \\
 &> z - c_9(n-m)
 \end{aligned}$$

Now, on supposing that $z \geq \frac{3}{2} + c_9(n-m)$ we may use property (8) of the p -adic logarithm to obtain

$$\nu_p(\log_p \delta) + \nu_p(ms+r) > z - c_9(n-m).$$

If we assume that $n > 17.5 \log |\alpha| (\max\{\log |2a\alpha|, \log |2b\beta|\} + 0.24)$ we get by a very crude estimate

$$\begin{aligned} z &< \nu_p(\log_p \delta) + 2 \frac{\log n}{\log p} + c_9(n-m) \\ &< \left(\nu_p(\log_p \delta) + \frac{2}{\log p} + c_9 \right) (n-m) \log n \\ &:= c_{11}(n-m) \log n \end{aligned}$$

Note that

$$\nu_p(\log_p(\delta)) = \nu_p \left(\frac{1}{s} \log_p \left(\frac{\alpha}{\beta} \right) \right) \leq \nu_p \left(\log_p \left(\frac{\alpha}{\beta} \right) \right).$$

Therefore we may assume that $\frac{b(\beta^{n-m}+1)}{a(\alpha^{n-m}+1)}$ and α/β are multiplicatively independent. However, before applying Corollary 1, we have to ensure that

$$\nu_p \left(\frac{b(\beta^{n-m}+1)}{a(\alpha^{n-m}+1)} \right) = \nu_p \left(\frac{\beta}{\alpha} \right) = 0.$$

Let us assume for the moment that one of the p -adic valuations is not zero. But this would imply that the p -adic valuation of the left hand side of (12) is zero, hence $z = 0$. Therefore we may apply Corollary 1 and we obtain

$$(15) \quad \nu_p(\Lambda) \leq C_1(p) \max\{\log |\alpha|, \log p\} c_{10}(n-m) \max\{\log n, 10 \log p, 10\}^2.$$

Comparing upper and lower bounds of $\nu_p(\Lambda)$ i.e. inequalities (13) and (15) yields

$$C_1(p) \max\{\log |\alpha|, \log p\} c_{10}(n-m) (\log n)^2 > z - c_9(n-m)$$

or by solving for z followed by a crude estimate we obtain our upper bound for z :

$$z < (C_1(p) \max\{2 \log |\alpha|, \log p\} c_{10} + c_9) (n-m) (\log n)^2 := c_8(n-m) (\log n)^2.$$

□

4. THE CASE $n = m$

Before we continue with the main line of the proof of Theorem 1 let us consider the special case $n = m$, i.e. the Diophantine equation

$$2u_n = wp_1^{z_1} \cdots p_s^{z_s}.$$

Of course this equation has only a solution if w is even or one of the primes, say $p_1 = 2$. Thus we are reduced to consider Diophantine equations of the type

$$(16) \quad u_n = w' p_1^{z_1} \cdots p_s^{z_s}$$

Let us note that this type of equation has been considered by Pethő and de Weger [12]. They gave a practical method to solve this equation completely for a given binary, non-degenerate sequence u_n with discriminant $\Delta > 0$, non-zero integer w and primes p_1, \dots, p_s by using p -adic techniques. In particular, they use linear forms in p -adic logarithms and a p -adic version of the Baker-Davenport method [1]. We also want to refer to the paper of Mignotte and Tzanakis [11], where also the case of non-degenerate recurrences of arbitrary order is discussed.

Proposition 2 (Pethő and de Weger [12]). *Let n, z_1, \dots, z_s be a solution to (16), then there exist explicitly computable constants $c_{12,i}$ with $1 \leq i \leq s$ and c_{13} such that $z_i < c_{12,i}$ for all $1 \leq i \leq s$ and $n < c_{13}$.*

The constant c_{13} has been stated explicitly by Pethő and de Weger [12, Theorem 4.1]. However they used a version due to Schinzel [13] for lower bounds in linear forms in p -adic logarithms that has been succeeded by newer developments e.g. the results due to Yu [16, 17] and in particular by Bugeaud and Laurent [6]. For the sake of completeness and with respect to the newer developments we give a proof of Proposition 2.

Proof. Since the proof due to Pethő and de Weger [12] and the similarity to the proof of Proposition 1 we only sketch the proof. As in the proof of Proposition 1 we fix an index i and drop it for the rest of the proof. We may assume without loss of generality that $\nu_p(\alpha) = \nu_p(b/a) = \nu_p(\beta/\alpha) = 0$. We rewrite equation (16) and obtain

$$(17) \quad \frac{b}{a} \cdot \left(\frac{\beta}{\alpha}\right)^n - 1 = w'(\alpha - \beta)a^{-1}\alpha^{-n}p_1^{z_1} \cdots p_s^{z_s}.$$

Let us assume for the moment that $\frac{b}{a}$ and $\frac{\beta}{\alpha}$ are multiplicatively independent and that $n > \max\{p^{10}, e^{10}\}$. Let us put $\Lambda = \frac{b}{a} \cdot \left(\frac{\beta}{\alpha}\right)^n - 1$. Estimating the p -adic valuation on the right hand side of (17) yields

$$\nu_p(\Lambda) \geq z - \frac{\log_* |a|}{\log p}$$

and an application of Corollary 1 yields

$$\nu_p(\Lambda) \leq C_1(p)h' \left(\frac{b}{a}\right)h' \left(\frac{\beta}{\alpha}\right) \max\{\log n, 10 \log p, 10\}^2,$$

with

$$h' \left(\frac{b}{a}\right) \leq \max\{\log |a|, \log |b|, \log p\}$$

and

$$h' \left(\frac{\beta}{\alpha}\right) \leq \max\{\log |\alpha|, \log |\beta|, \log p\}.$$

Thus we get

$$\begin{aligned} z &< \left(C_1(p)h' \left(\frac{b}{a}\right)h' \left(\frac{\beta}{\alpha}\right) + \frac{\log_* |a|}{\log p} \right) \max\{\log n, 10 \log p, 10\}^2 \\ &= c_{14} \max\{\log n, 10 \log p, 10\}^2. \end{aligned}$$

If we assume that $n > c_3$ we may apply Lemma 2 (iv) and obtain either an absolute bound for n or an inequality of the form $n < c_{15}(\log n)^2$ and Lemma 1 yields an absolute upper bound for n . In any case we obtain the Proposition.

We are left with the case that $\frac{a}{b}$ and $\frac{\alpha}{\beta}$ are multiplicatively dependent, i.e. there are co-prime integers s and r such that $\left(\frac{a}{b}\right)^s = \left(\frac{\alpha}{\beta}\right)^r$ and let $\delta = \left(\frac{\alpha}{\beta}\right)^{1/s}$. As in the proof of Proposition 1 we deduce $|s| < 4.2 \log |\alpha|$ and by a similar computation we obtain $|r| < 17.5 \log |\alpha| \max\{\log |a|, \log |b|, 1\}$. Hence we obtain from (17) the inequality

$$\nu_p(\delta^{-ns-r} - 1) \geq z - \frac{\log_* |a|}{\log p}.$$

Now on supposing $z \geq \frac{3}{2} + \frac{\log_* |a|}{\log p}$ we may use the properties of the p -adic logarithms (e.g. see formula (8)) and we obtain

$$z < \nu_p(\log_p \delta) + \frac{\log(4.2n \log |\alpha| + 17.5 \log |\alpha| \max\{\log |a|, \log |b|, 1\}) + \log_* |a|}{\log p}.$$

Assuming that

$$n > 17.5 \log |\alpha| \max\{\log |a|, \log |b|, 1\}$$

we obtain $z < \left(\nu_p(\log_p \delta) + \frac{2}{\log p} + \frac{\log_* |a|}{\log p} \right) \log n$, and by Lemma 2 (iv) and the assumption $n > c_3$ we obtain

$$n < \frac{\sum_{i=1}^s (\nu_{p_i}(\log_{p_i} \delta) \log p_i + (2 + \log_* |a|))}{\log |\alpha|} \log n + c_5 = c_{16} \log n + c_5$$

and once again applying Lemma 1 yields an upper bound c_{13} for n (for an explicit determination see Section 10). \square

5. THE CASE $n > m$

In order to solve the case $n > m$, we have to use lower bounds for linear forms in complex logarithms. The currently best result suitable for our purpose is the following theorem due to Matveev [10].

Theorem 3 (Matveev [10]). *Denote by η_1, \dots, η_n algebraic numbers, not 0 or 1, by $\log \eta_1, \dots, \log \eta_n$ determinations of their logarithms, by D the degree over \mathbb{Q} of the number field $L = \mathbb{Q}(\eta_1, \dots, \eta_n)$, and by b_1, \dots, b_n rational integers. Define $B' = \max\{|b_1|, \dots, |b_n|\}$, and $A_i = \max\{Dh(\eta_i), |\log \eta_i|, 0.16\}$ ($1 \leq i \leq n$), where $h(\eta)$ denotes the absolute logarithmic Weil height of η . Assume that the number $\Lambda = b_1 \log \eta_1 + \dots + b_n \log \eta_n$ does not vanish. Then*

$$\log |\Lambda| \geq -C(n, \varkappa) D^2 A_1 \cdots A_n \log(eD) \log(eB'),$$

where $\varkappa = 1$ if $\mathbb{K} \subset \mathbb{R}$ and $\varkappa = 2$ otherwise and

$$C(n, \varkappa) = \min \left\{ \frac{1}{\varkappa} \left(\frac{1}{2} en \right)^\varkappa 30^{n+3} n^{3.5}, 2^{6n+20} \right\}.$$

Since the field $K = \mathbb{Q}(\sqrt{\Delta})$ of our interest is real and of degree at most 2 and all the α_i will be positive we obtain

$$(18) \quad \log |\Lambda| \geq -C_2(n) h(\eta_1) \cdots h(\eta_n) \log B',$$

where

$$C_2(n) = 2.31 \cdot 60^{n+3} n^{4.5}$$

provided that $B' \geq 3$. Note that since Lemma 3 we can choose $A_i = 2h(\eta_i)$.

The main purpose of this section is to find absolute upper bounds for n and the exponents z_i with $1 \leq i \leq s$. For technical reasons we will assume that $n - m$ is not too small. In particular, we may assume that

$$n - m > \frac{\log \left(2 \left(1 + \frac{2|b|}{|a|} \right) \right)}{\log \left(\min \left\{ \frac{|\alpha|}{|\beta|}, |\alpha| \right\} \right)} = c_{17}.$$

Indeed, due to Proposition 1 the case that $n - m \leq c_{17}$ immediately implies $n < c_7(n - m)(\log n)^2 < c_7 c_{17} (\log n)^2$. Thus by Lemma 1 we get that $n \leq 4c_7 c_{17} \log(4c_7 c_{17})^2$.

Hence we assume for the rest of this section that $n - m > c_{17}$. Let us rewrite equation (1) as

$$(19) \quad |p_1^{z_1} \cdots p_s^{z_s} w(\alpha - \beta) a^{-1} \alpha^{-n} - 1| = \frac{|-b\beta^n + a\alpha^m - b\beta^m|}{|a\alpha^n|}$$

Let us estimate the right hand side of (19). We obtain

$$\begin{aligned} \frac{|-b\beta^n + a\alpha^m - b\beta^m|}{|a\alpha^n|} &\leq \left| \frac{2b \max\{\beta^n, \beta^m\}}{a\alpha^n} \right| + |\alpha|^{m-n} \\ &\leq \frac{2|b|}{|a|} \max \left\{ \frac{|\beta|^n}{|\alpha|^n}, \frac{|\beta|^m}{|\alpha|^n} \right\} + |\alpha|^{m-n} \\ &\leq \frac{2|b|}{|a|} \max \left\{ \left(\frac{|\alpha|}{|\beta|} \right)^{m-n}, |\alpha|^{m-n} \right\} + |\alpha|^{m-n} \\ &\leq \left(1 + \frac{2|b|}{|a|} \right) \max \left\{ \frac{|\beta|}{|\alpha|}, \frac{1}{|\alpha|} \right\}^{n-m}. \end{aligned}$$

Note that our assumption $n - m \geq c_{17}$ was chosen such that the right hand side of equation (19) is smaller than $1/2$. Since $|\log |x+1|| \leq 2|x|$ provided $0 \leq |x| \leq 1/2$ we obtain by taking logarithms on both sides of equation (19) the inequality

$$(20) \quad |\Lambda| := |z_1 \log p_1 + \cdots + z_s \log p_s + \log |\gamma| - n \log |\alpha| < \frac{2 \left(1 + \frac{2|b|}{|a|} \right)}{\min \left\{ \frac{|\alpha|}{|\beta|}, |\alpha| \right\}^{n-m}},$$

where $\gamma = \frac{w\sqrt{\Delta}}{a}$. In order to apply Matveev's Theorem we have to ensure that Λ does not vanish. However this is established by the following Lemma:

Lemma 4. *If $\Lambda = 0$ then there exists a constant c_{18} such that $n < c_{18}$.*

In order to complete the proof of Theorem 1 we postpone the proof of Lemma 4 to the next section. Now by Lemma 4 we may assume that Λ does not vanish and we may apply Matveev's Theorem. Before we apply Matveev's Theorem let us note that

$$B' = \max\{z_1, \dots, z_s, 1, n\} \leq n \frac{2 \log |\alpha|}{\log 2} < n^2,$$

provided that $n > \max\{c_2, \frac{2 \log |\alpha|}{\log 2}\}$. Thus by comparing the upper bound from inequality (20) with the lower bound of $|\Lambda|$ from Matveev's bound (18) we obtain the inequality

$$\begin{aligned} C_2(s+2) \log p_1 \cdots \log p_s \log |\alpha| h(\gamma) 2 \log n > \\ (n-m) \log \left(\max \left\{ \frac{|\alpha|}{|\beta|}, |\alpha| \right\} \right) - \log \left(1 + \frac{2|b|}{|a|} \right), \end{aligned}$$

i.e. there exists a constant c_{19} (see Section 10 for an explicit form) such that $n - m < c_{19} \log n$. In combination with Proposition 1 we obtain $n < c_7 c_{19} (\log n)^3$, provided that $n > c_6$. Once again using Lemma 1 yields an upper bound c_{20} for n and Lemma 2 yields an upper bound $z_i < c_{21,i}$ for each $1 \leq i \leq s$.

With the notations from above in force we have proved so far the following proposition, which clearly implies Theorem 1:

Proposition 3. *Under the assumptions of Theorem 1 we have*

$$\max\{n, m\} < c_{20} \quad \text{and} \quad z_i < c_{21,i}$$

for $i = 1, \dots, s$.

6. PROOF OF LEMMA 4

First, let us note that $\Lambda = 0$ implies that

$$(21) \quad b\beta^n - a\alpha^m + b\beta^m = 0.$$

In the case that the $\Delta > 0$ is not a perfect square α and β respectively a and b are algebraic conjugate and we obtain by conjugating the left side of equation (21)

$$-a\alpha^n + b\beta^m - a\alpha^m = 0$$

which yields together with the original equation (21) the relation $-b\beta^n = a\alpha^n$. Therefore $n < \frac{\log |b/a|}{\log |\alpha/\beta|} = c_{22}$.

Therefore we may assume that Δ is a perfect square and that α, β, a, b are all rational integers. Assume for the moment that $|\beta| = 1$. Then (21) turns into $a\alpha^m = 0$ or $a\alpha^m = 2b$. The first case would imply that u_n is degenerate and the second case has been excluded. Thus we may assume that $|\beta| \geq 2$.

Under these assumptions we obtain from (21) by dividing through $b\beta^n$ the inequality

$$\left| \frac{|a|}{|b|} \frac{|\alpha|^m}{|\beta|^n} - 1 \right| \leq |\beta|^{m-n}$$

and by taking logarithms we get

$$(22) \quad \Lambda := \log \left| \frac{a}{b} \right| + m \log |\alpha| - n \log |\beta| < \frac{2}{|\beta|^{n-m}}$$

Let us assume for the moment that $\Lambda = 0$. Since a, b, α, β are integers $\Lambda = 0$ implies that $a\alpha^m = \pm b\beta^n$ and in combination with equation (21) this leads either to $b\beta^m = 0$ or $\beta^{n-m} = -\frac{1}{2}$. But, both cases are contradictions in view of $|\beta| \geq 2$ and $n > m$.

Hence we may suppose that $\Lambda \neq 0$ and therefore we may apply Matveev's Theorem 3 to the left side of inequality (22) and obtain

$$\log |\Lambda| > -C_2(3) \max\{0.16, \log_* \max\{|a|, |b|\}\} \log |\alpha| \log |\beta| \log n.$$

On the other hand we have

$$\log |\Lambda| < \log 2 - (n - m) \log |\beta|.$$

Hence we get that $n - m < c_{23} \log n$ (see Section 10 for an explicit determination of c_{23}). Combining this with the results of Proposition 1 we get $n < c_7 c_{23} (\log n)^3$ provided that $n > c_6$. Thus we find an absolute upper bound c_{18} for n by applying Lemma 1. Let us note that in Section 8 we explain in more detail how to handle this case in practice.

7. REDUCTION OF OUR BOUNDS

This section is devoted to the problem of reducing the rather large bounds obtained by Theorem 1 and Proposition 3 respectively. In this paper we will make use of the LLL-algorithm due to Lenstra, Lenstra and Lovász [8] to reduce our upper bounds for $n - m$ and z_1, \dots, z_s . In the p -adic case we use instead of approximation lattices an idea due to Pethő and de Weger [12, Algorithm A].

7.1. Real approximation lattices. Let us start with gathering some basic facts on LLL-reduced bases and approximation lattices. Therefore let $\mathcal{L} \subseteq \mathbb{R}^k$ be a k -dimensional lattice with LLL-reduced basis b_1, \dots, b_k and let B be the matrix with columns b_1, \dots, b_k . Moreover, we denote by b_1^*, \dots, b_k^* the orthogonal basis of \mathbb{R}^k which we obtain by applying the Gram-Schmidt process to the basis b_1, \dots, b_k . In particular, we have that

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j^*, \quad \mu_{i,j} = \frac{\langle b_i, b_j^* \rangle}{\langle b_j^*, b_j^* \rangle}.$$

Further, let us define

$$l(\mathcal{L}, y) = \begin{cases} \min_{x \in \mathcal{L}} \{\|x - y\|\}, & y \notin \mathcal{L} \\ \min_{0 \neq x \in \mathcal{L}} \{\|x\|\}, & y \in \mathcal{L}, \end{cases}$$

where $\|\cdot\|$ denotes the euclidian norm on \mathbb{R}^k . It is well known, that by applying the LLL algorithm it is possible to give in a polynomial time a lower bound for $l(\mathcal{L}, y) \geq \tilde{c}_1$ (see e.g. [14, Section 5.4]).

Lemma 5. *Let $y \in \mathbb{R}^k$, $z = B^{-1}y$ and if $y \notin \mathcal{L}$ let i_0 be the largest index such that $z_{i_0} \neq 0$. Put $\sigma = \{z_{i_0}\}$, where $\{\cdot\}$ denotes the distance to the nearest integer, and in case that $y \in \mathcal{L}$ we put $\sigma = 1$. Moreover, let*

$$\tilde{c}_2 = \max_{1 \leq j \leq k} \left\{ \frac{\|b_1\|^2}{\|b_j^*\|^2} \right\}.$$

Then we have

$$l(\mathcal{L}, y)^2 \geq \tilde{c}_2^{-1} \sigma \|b_1\|^2 = \tilde{c}_1.$$

In our applications suppose we are given $\eta_0, \eta_1, \dots, \eta_k$ real numbers linearly independent over \mathbb{Q} and two positive constants \tilde{c}_3, \tilde{c}_4 such that

$$(23) \quad |\eta_0 + x_1 \eta_1 + \dots + x_k \eta_k| \leq \tilde{c}_3 \exp(-\tilde{c}_4 H),$$

where the integers x_i with $1 \leq i \leq k$ are bounded by $|x_i| \leq X_i$ with X_i given upper bounds for all $1 \leq i \leq k$. Set $X_0 = \max_{1 \leq i \leq k} \{X_i\}$. The basic idea in such a situation, due to de Weger [7], is to approximate the linear form (23) by an approximation lattice. Namely, we consider the lattice \mathcal{L} generated by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ \lfloor C\eta_1 \rfloor & \lfloor C\eta_2 \rfloor & \dots & \lfloor C\eta_{k-1} \rfloor & \lfloor C\eta_k \rfloor \end{pmatrix}$$

where C is a large constant usually of the size about X_0^k . Let us assume that we have an LLL-reduced basis b_1, \dots, b_k of \mathcal{L} and that we have a lower bound $l(\mathcal{L}, y) \geq \tilde{c}_1$ with $y = (0, 0, \dots, -\lfloor C\eta_0 \rfloor)$. Then we have with these notations the following Lemma concerning inequality (23) (c.f. [14, Lemma VI.1]):

Lemma 6. Assume that $S = \sum_{i=1}^{k-1} X_i^2$ and $T = \frac{1+\sum_{i=1}^k X_i}{2}$. If $\tilde{c}_1^2 \geq T^2 + S$, then we have either $x_1 = x_2 = \cdots = x_{k-1} = 0$ and $x_k = -\frac{[C\eta_0]}{[C\eta_k]}$ or

$$(24) \quad H \leq \frac{1}{\tilde{c}_4} \left(\log(C\tilde{c}_3) - \log \left(\sqrt{\tilde{c}_1^2 - S - T} \right) \right).$$

We will apply Lemma 6 to inequality (20), i.e. $k = s + 1$ and $x_i = z_i$ for $i = 1, \dots, s$ and $x_{s+1} = n$. For the coefficients η_i with $0 \leq i \leq s + 1$ we take $\eta_0 = \log \gamma$, $\eta_i = \log p_i$ for $1 \leq i \leq s$ and $\eta_{s+1} = \log |\alpha|$. Furthermore we have $\tilde{c}_3 = 2 \left(1 + \frac{2|b|}{|a|} \right)$ and $\tilde{c}_4 = \log \left(\min \left\{ \frac{|\alpha|}{|\beta|}, |\alpha| \right\} \right)$.

For sure η_1, \dots, η_s are linearly independent over \mathbb{Q} and it is easy to find a multiplicative dependence of γ and/or α of the p_i with $i = 1, \dots, s$. However, let us assume for the moment that $\log |\gamma|, \log p_1, \dots, \log p_s$ and $\log |\alpha|$ are linearly independent over \mathbb{Q} (a similar argument holds if this is not the case). Let us also assume that C was chosen large enough such that

$$\tilde{c}_1^2 > \sum_{i=1}^s c_{21,i}^2 + \left(\frac{\sum_{i=1}^s c_{21,i} + c_{20}}{2} \right)^2 = T^2 + S.$$

Then we get by Lemma 6 either a new bound for $H = n - m$ or $z_1 = \dots = z_s = 0$ and our Diophantine equation (1) reduces to $u_n + u_m = w$ which can be easily solved by brute force if w is not too large.

Let us note that a new bound for $n - m$ immediately yields new upper bounds for z_i with $1 \leq i \leq s$ and n by Proposition 1. We can apply the same trick once again with these new bounds and obtain again a further reduction of the bounds for z_i with $1 \leq i \leq s$ and n . We can repeat this as long as our reduction method yields smaller upper bounds for $n - m$.

7.2. The p -adic reduction method. In the p -adic case it is also possible to use so-called p -adic approximation lattices (see e.g. [14, Section 5.6]). However we are only interested in a very special case namely in the situation appearing in inequality (12) under the assumption that $n - m = t$ is a fixed (small) integer. The following is based on an idea due to Pethő and de Weger [12, Algorithm A]. We reproduce their idea and fit it into our framework.

Let us fix an index i with $1 \leq i \leq s$. In order to avoid an overloaded notation we drop the index i for the rest of this section. Furthermore let us assume that N and Z are given upper bounds for n and z respectively. Let us consider the p -adic valuation of the left and right side of (12). Then we obtain (as in Section 3)

$$\nu_p \left(\tau(t) \left(\frac{\beta}{\alpha} \right)^m - 1 \right) = z - \nu_p(a(\alpha^t + 1)) + \nu_p(\alpha - \beta) = z - z_0,$$

where

$$\tau(t) = \frac{b(\beta^t + 1)}{a(\alpha^t + 1)}$$

and z_0 is easily computable for a fixed t . Let us assume that $z \geq 3/2 + z_0 = \tilde{c}_5$, then we may take p -adic logarithms and obtain

$$(25) \quad \nu_p \left(\log_p(\tau(t)) - m \log_p(\alpha/\beta) \right) = z - z_0.$$

We distinguish now between two cases whether $\log_p(\tau(t)) = 0$ or not.

Let us discuss the rather unlikely case that $\log_p(\tau(t)) = 0$ first. In this case equation (25) turns into

$$\nu_p(m \log_p(\alpha/\beta)) = \nu_p(m) + \nu_p(\log_p(\alpha/\beta)) = z - z_0,$$

i.e.

$$z < \frac{\log m}{\log p} + \nu_p(\log_p(\alpha/\beta)) + z_0 < \frac{\log N}{\log p} + \nu_p(\log_p(\alpha/\beta)) + z_0 = \tilde{c}_6$$

and $\max\{\tilde{c}_5, \tilde{c}_6\}$ is a new upper bound for z .

Let us turn to the case that $\log_p(\tau(t)) \neq 0$. Since α and β are conjugate in $\mathbb{Q}(\sqrt{\Delta}) \subset \mathbb{Q}_p(\sqrt{\Delta})$ also $\log_p(\alpha)$ and $\log_p(\beta)$ are conjugate, hence $\log_p(\alpha/\beta) = \log_p(\alpha) - \log_p(\beta) \in \sqrt{\Delta}\mathbb{Q}_p$. Similarly we get $\log_p(\tau(t)) \in \sqrt{\Delta}\mathbb{Q}_p$, since $\tau(t)$ is the quotient of conjugates. In particular, we get

$$\zeta = \log_p(\tau(t))/\log_p(\alpha/\beta) = u_0 + u_1p + u_2p^2 + \dots \in \mathbb{Q}_p.$$

Let r be the smallest possible exponent such that $p^r > N$ and let $0 \leq m_0 < p^r$ be the unique integer such that $m_0 \equiv \zeta \pmod{p^r}$. Moreover, let R be the smallest index $\geq r$ such that $u_R \neq 0$, if such an index exists. Then we get

$$\begin{aligned} z - z_0 &= \nu_p(\log_p(\tau(t)) - m \log_p(\alpha/\beta)) \leq \nu_p(\log_p(\tau(t)) - m_0 \log_p(\alpha/\beta)) \\ &= \nu_p(\log_p(\tau(t)) + \nu_p\left(1 - (\zeta - u_R p^R - \dots) \frac{\log_p(\alpha/\beta)}{\log_p(\tau(t))}\right)) \\ &= \nu_p(\log_p(\tau(t)) + R + \nu_p\left(\frac{\log_p(\alpha/\beta)}{\log_p(\tau(t))}\right)). \end{aligned}$$

Therefore we get a new upper bound for z namely

$$z \leq \nu_p(\log_p(\tau(t)) + R + \nu_p\left(\frac{\log_p(\alpha/\beta)}{\log_p(\tau(t))}\right)).$$

Let us discuss the case, in which R does not exist. In this case we would obtain that $\zeta = m_0$ is an integer, hence

$$\log_p(\tau(t)) = m_0 \log_p(\alpha/\beta),$$

and (25) turns into

$$\nu_p(\log_p(\tau(t)) - m \log_p(\alpha/\beta)) = \nu_p((m - m_0) \log_p(\alpha/\beta)) = z - z_0.$$

Therefore we are in a similar situation as in the case that $\log_p(\tau(t)) = 0$ and we also get in this case a new upper bound for z .

In any case we get for each index i with $1 \leq i \leq s$ a new upper bound z_i and therefore by Lemma 2 (iv) a new upper bound for n . We can repeat the procedure with these new, small upper bounds as long as we get smaller upper bounds for n .

8. THE ALGORITHM

At this point the inclined reader may have already a fairly well idea how to solve Diophantine equations of type (1) in practice. However, let us summarize the key steps and give some comments on the practical implementation. The algorithm to solve Diophantine equation (1) can be splitted up into six key steps:

Step I - First upper bounds for n and z_i for all $1 \leq i \leq s$: Compute one after the other the constants c_1, \dots until you have found the upper bounds $n < c_{20}$ and $z_i < c_{21,i}$ with $1 \leq i \leq s$. Using the explicit determination of the constants given in Section 10 this is straight forward.

Step II - The case that $n = m$: If w and all p_i with $1 \leq i \leq s$ are odd, this case has no solution. Otherwise in case that w is even we replace w by $w/2$ or if $p_1 = 2$ we replace z_1 by $z_1 - 1$ and compute upper bounds for $n < c_{13}$ and $z_i < c_{12,i}$ for $1 \leq i \leq s$. With these upper bounds we use our p -adic reduction procedure from Section 7.2 with $t = 1$ and $\tau(t) = b/a$ and obtain a small bound for n , which is in most cases small enough to performe a brute force search.

Step III - The case that $b\beta^n - a\alpha^m + b\beta^m = 0$: If Δ is not a perfect square or $\beta = \pm 1$ there are no solutions in this case and we may omitt this step. Otherwise we may assume that α, β, a and b are rational integers and that $|\beta| \geq 2$.

In the case that $\beta \nmid \alpha$ we obtain the equation

$$\frac{a}{b} \left(\frac{\alpha}{\beta} \right)^m = \beta^{n-m} + 1$$

We may assume that $\alpha/\beta = P/Q$ with $Q > 1$ and P, Q are coprime integers. Then $m \leq \log Q / \log a$ since otherwise the left hand side cannot be an integer. This presumably small bound for m yields also a small bound for n and a brute force search will resolve this case.

Therefore we may assume that $\alpha/\beta = P$ is an integer and let us assume that p is the largest prime factor dividing P and let us assume for the moment that $m > 1$. Then we have that

$$\begin{aligned} m - \nu_p(b) &= \nu_p(\beta^{n-m} - 1) \\ &= \nu_p(\log_p(\beta)) + \nu_p(n - m) \\ &< \nu_p(\log_p(\beta)) + \log(c_{23} \log c_{18}). \end{aligned}$$

Usually this will yield a reasonable small bound for m and n to perform a brute force search. If the new upper bound $\max\{n, m\} < \tilde{C}$ is still large one may use once again the above inequality to obtain

$$\begin{aligned} m - \nu_p(b) &= \nu_p(\log_p(\beta)) + \nu_p(n - m) \\ &< \nu_p(\log_p(\beta)) + \frac{\log(\tilde{C})}{\log p}. \end{aligned}$$

This may be repeated until no further improvement on the bound for $\max\{n, m\}$ is possible and one has to perform a brute force search for possible solutions to Diophantine equation (1).

Step IV - First reduction of n : Perform the reduction step described in Section 7.1 to obtain upper bounds $n < N$, $z_i < Z_i$ for $1 \leq i \leq s$ and $n - m \leq T_0$.

Step V - Second reduction of n : Perform the reduction step described in Section 7.2 for each $1 \leq t \leq T_0$ and obtain for each t upper bounds $n < \tilde{N}_t$ and $z_i < \tilde{Z}_{i,t}$ for $1 \leq i \leq s$. Thus we get new upper bounds for $n < \tilde{N} = \max_{1 \leq t \leq T_0} \{\tilde{N}_t\}$ and

$z_i < \tilde{Z}_i = \max_{1 \leq t \leq T_0} \{\tilde{Z}_{i,t}\}$ for $1 \leq i \leq s$.

Step VI - Brute force search: The upper bounds \tilde{N} and \tilde{Z}_i for $1 \leq i \leq s$ obtained in Step V are usually rather small. However replacing c_{23} by \tilde{N} and $c_{22,i}$ by \tilde{Z}_i for $1 \leq i \leq s$ we can go back to Step IV and after performing the reduction steps IV and V again maybe we obtain sharper bounds for n and z_i with $1 \leq i \leq s$. This can be repeated until now further improvement is possible. And we have to check the remaining cases by a brute force search. For instance we compute for all $1 \leq n \leq m \leq \tilde{N}$ the values of $u_n + u_m$ and write them into a list \mathcal{L} . For each element from the list \mathcal{L} we perform a trial division including the primes p_1, \dots, p_s . If \tilde{N} and $P = \max\{p_1, \dots, p_s\}$ are not unusually large, say $\tilde{N}, P < 10000$ this brute force search can be done within a reasonable time (see Section 9).

9. AN EXAMPLE

In this section we illustrate our algorithm by two examples. We completely solve Diophantine equation (1) in the case that $w = 1$, p_1, \dots, p_{46} are all primes smaller than 200 and u_n is the Fibonacci sequence or the Lucas sequence, respectively. We have the following theorem:

Theorem 4. (i) Let $\{F_n\}_{n \geq 0}$ be the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Consider the equation

$$(26) \quad F_n + F_m = 2^{z_1} 3^{z_2} \dots 199^{z_{46}}$$

in non-negative integer unknowns n, m, z_1, \dots, z_{46} with $n \geq m$. Then there are 325 solutions $(n, m, z_1, \dots, z_{46})$ and there exists no solution with $\max\{n, m\} > 59$.

(ii) Let $\{L_n\}_{n \geq 0}$ be the Lucas sequence defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Consider the equation

$$(27) \quad L_n + L_m = 2^{z_1} 3^{z_2} \dots 199^{z_{46}}$$

in non-negative integer unknowns n, m, z_1, \dots, z_{46} with $n \geq m$. Then there are 284 solutions $(n, m, z_1, \dots, z_{46})$ and there exists no solution with $\max\{n, m\} > 63$.

We refrain from giving a list of all 325 and 284 solutions to Diophantine equations (26) and (27), respectively. Let us note that a brute force computer search for all solutions to (26) and (27) with $m \leq n \leq 59$ and $m \leq n \leq 63$ respectively is a matter of a few seconds on a usual PC.

However looking through the solutions to Diophantine equations (26) and (27) respectively we noticed the following interesting facts.

- $z_1 \leq 8$ and $z_2 \leq 6$ in both cases.
- In the case of equation (26) we have that $z_3, z_5 \leq 3$ and for all other indices $\neq 1, 2, 3, 5$ we have that $z_i \leq 2$.
- In the case of equation (27) we have that $z_3 \leq 3$ and for all other indices $i \geq 4$ we have that $z_i \leq 2$.
- For all solutions to equation (26) the exponents of 79, 83, 131, 139, 163 and 167 are always zero.
- For all solutions to equation (27) the exponent of 73, 149, 157, 173, 181, 191, 193 and 197 are always zero.

Step I - First upper bounds for n and z_i for all $1 \leq i \leq s$: By using the explicit determination of the constants given in Section 10 we obtain by a straightforward computation that

Step II - The case that $n = m$: It is clear, that in this case we may suppose that $z_1 \geq 1$, since otherwise we do not have a solution. Now, we replace z_1 with $z_1 - 1$ and compute the initial upper bounds $n < c_{13}$ and $z_i < c_{12,i}$ ($1 \leq i \leq 46$) and get $\max\{n, z_i\} < N := 1.4 \cdot 10^{23}$. For every $p = p_i$ ($1 \leq i \leq 46$) we perform the p -adic reduction procedure described in Section 7.2 with $t = 1$ and $\tau(t) = b/a = 1$. Since $\alpha - \beta = \sqrt{5}$ and $a = 1$ we obtain that $z_0 = \nu_p(a(\alpha^t + 1)) - \nu_p(\alpha - \beta) \leq 1$ and $\log_p(\tau(t)) = \log_p(1) = 0$ holds for every $p = p_i$ ($1 \leq i \leq 46$). Further, $\nu_p(\log_p(\alpha/\beta)) \leq 2$ and hence we get that

Note that the values $Z_{0,i}$ form a vector

Since in our case $c_5 < 6$, we infer by (iv) of Lemma 2 that

By repeating the above reduction step once more, now with $N = 6100$, we obtain $n \leq 1771$.

Step III - The case that $b\beta^n - a\alpha^m + b\beta^m = 0$: One can easily see that this case cannot occur if $\{u_n\}_{n \geq 0} = \{F_n\}_{n \geq 0}$.

Step IV - First reduction of n : We may suppose that $n > m$. We apply the reduction method described in Section 7.1 with the initial upper bound provided by Step I. Namely $\max\{n, z_i\} < X_{00} := 2.6 \cdot 10^{117}$ and choose $C := 10^{6300}$. Further, put $\tilde{c}_3 = 6$, $\tilde{c}_4 = \log \frac{1+\sqrt{5}}{2}$, $k = 47$, $x_i = z_i$ with $1 \leq i \leq 46$, $x_{47} = n$ and $\eta_0 = \log(\sqrt{5})$, $\eta_i = \log p_i$ with $1 \leq i \leq 46$. After performing the LLL algorithm we found that $(\tilde{c}_2)^{-1} := 1/25000$ and $\sigma := 0.49$ is an appropriate choice. Thus, we may apply Lemma 6 and we obtain a reduced bound $n - m \leq T_1 = 30000$.

Inserting this bound for $n - m$ into Proposition 1 we are able to derive a new, better bound for $\max\{n, z_i\}$. An easy calculation shows that $\max\{n, z_i\} <$

$X_{01} := 199^{10}$ is an appropriate choice. By repeating once more the above procedure, with $X_{01} := 199^{10}$, $C := 10^{1280}$, $(\tilde{c}_2)^{-1} := 1/81550$ and $\sigma := 0.49$ we obtain $n - m \leq T_2 = 6010$.

Step V - Second reduction of n : We perform the reduction step described in Section 7.2 for each $1 \leq t \leq T_2 = 6010$. Since $\max\{n, z_i\} < X_{01} := 199^{10}$ we apply the reduction step described in Section 7.2 for each prime $p = p_i$ with $1 \leq i \leq 46$ and for each $1 \leq t \leq T_2 = 6010$. In particular, we obtain that $z = z_i \leq Z_{1,i}$ with $1 \leq i \leq 46$, where the $Z_{1,i}$ form a vector

$$Z_1 = (92, 57, 41, 37, 24, 27, 25, 19, 22, 20, 17, 21, 18, 20, 18, 20, 16, 16, 17, 14, 17, 14, 15, 15, 15, 13, 16, 16, 15, 17, 14, 13, 14, 14, 13, 12, 14, 14, 14, 14, 12, 13, 13, 14, 15, 14).$$

Since in our case $c_5 < 6$ we infer by (iv) of Lemma 2 that

$$n < \frac{\sum_{i=1}^{46} Z_{1,i}(\log p_i)}{\log(\frac{1+\sqrt{5}}{2})} + 6 \leq 6600.$$

By repeating the above reduction step once more, now with $X_{02} = 6600$, we obtain $n \leq 2300$.

Step VI - Brute force search: Now, we have proved that $m \leq n \leq 2300$, a value which is small enough to provide a brute force search. Namely, we used trial division with primes up to 200 for each quantity $F_n + F_m$. We found 324 solutions.

Finally, we note that the total computational time of our algorithm for the sequence $\{F_n\}_{n \geq 0}$ was about four and a half hours on a computer with an Intel Core 5 M3230 processor. The most time consuming step was the first LLL-algorithm in Step IV which took almost four hours. \square

10. APPENDIX - EXPLICIT BOUNDS

Let us denote by $P = \max_{1 \leq i \leq s} \{p_i\}$.

Constants appearing in Lemma 2:

$$c_1 = 2 \frac{|a| + |b|}{\sqrt{\Delta}}, \quad c_2 = \frac{\log_* \frac{c_1}{|w|}}{\log |\alpha|}, \quad c_3 = \max \left\{ \frac{\log_* \frac{4|b|\varphi}{|a|(\varphi-1)}}{\log |\alpha|}, \frac{\log_* \frac{4|b|\varphi}{|a|(\varphi-1)}}{\log \frac{|\alpha|}{|\beta|}} \right\},$$

$$c_4 = \frac{|a|(\varphi-1)}{2\varphi\sqrt{\Delta}}, \quad c_5 = \frac{\log \frac{|w|}{c_4}}{\log |\alpha|}$$

The constant $C_1(p)$ from Bugeaud and Laurent's lower bound of linear forms in two p -adic logarithms:

$$C_1(p) = \frac{947p^{f_p}}{\log^4 p}.$$

Constants appearing in Proposition 1 and its proof:

$$\begin{aligned}
 c_6 &= \max\{c_3, 17.5 \log |\alpha| (\max\{\log |2a\alpha|, \log |2b\beta|\} + 0.24), P^{10}, e^{10}\}, \\
 c_9 &= \frac{\log |2a\alpha|}{\log p}, \\
 c_{10} &= \max\{\log |2a\alpha|, \log |2b\beta|, \log p\}, \\
 c_{11} &= \nu_p(\log_p(\alpha/\beta)) + \frac{2}{\log p} + c_9, \\
 c_{8,i} &= \max\{C_1(p_i) \max\{\log |\alpha|, \log p_i\} c_{10} + c_9, c_{11}\}, \\
 c_7 &= \frac{\sum_{i=1}^s c_{8,i} \log p_i}{\log |\alpha|} + c_5,
 \end{aligned}$$

where $P = \max_{1 \leq i \leq s} \{p_i\}$.

Constants appearing in Proposition 2 and its proof:

$$\begin{aligned}
 c_{14} &= C_1(p) h' \left(\frac{b}{a} \right) h' \left(\frac{\beta}{\alpha} + \frac{\log_* |a|}{\log p} \right) \\
 c_{15} &= \frac{\sum_{i=1}^s c_{14} \log p_i}{\log |\alpha|} + c_5, \\
 c_{16} &= \frac{\sum_{i=1}^s (\nu_{p_i}(\log_{p_i}(\alpha/\beta)) \log p_i + (2 + \log_* |a|))}{\log |\alpha|} \\
 c_{13} &= \max\{4c_{15} \log(4c_{15})^2, 2(c_5 + \log c_{16}), c_3, c_2, P^{10}, e^{10}\}, \\
 c_{12,i} &= \frac{2 \log |\alpha|}{\log p_i} c_{13}.
 \end{aligned}$$

where

$$h' \left(\frac{a}{b} \right) \leq \max\{\log |a|, \log |b|, \log P\}, \quad \text{and} \quad h' \left(\frac{\beta}{\alpha} \right) \leq \max\{\log |\alpha|, \log |\beta|, \log P\}.$$

The constant $C_2(n)$ from Matveev's lower bound of linear forms in complex logarithms:

$$C_2(n) = 2.31 \cdot 60^{n+3} n^{4.5}.$$

Constants appearing in the proof of Lemma 4

$$\begin{aligned}
 c_{22} &= \frac{\log |b/a|}{\log |\alpha/\beta|}, \\
 c_{23} &= 1.52 \cdot 10^{13} \max\{0.16, \log_* \max\{|a|, |b|\}\} \log |\alpha|, \\
 c_{18} &= \max\{8c_7 c_{23} \log(27c_7 c_{23})^3, c_6\}.
 \end{aligned}$$

Constants appearing in the rest of Section 5:

$$\begin{aligned}
 c_{17} &= \frac{\log \left(2 \left(1 + \frac{2|b|}{|a|} \right) \right)}{\log \left(\min \left\{ \frac{|\alpha|}{|\beta|}, |\alpha| \right\} \right)}, \\
 c_{19} &= \frac{2C_2(s+2) \log p_1 \dots \log p_s h(w\sqrt{\Delta}a^{-1}) \log |\alpha| + \log \left(1 + \frac{2|b|}{|a|} \right)}{\log \left(\max \left\{ \frac{|\alpha|}{|\beta|}, |\alpha| \right\} \right)}, \\
 c_{20} &= \max \{ 8c_7c_{19} \log(27c_7c_{19})^3, c_{18}, c_6, c_2 \}, \\
 c_{21,i} &= \frac{2 \log |\alpha|}{\log p_i} c_{20}.
 \end{aligned}$$

REFERENCES

- [1] A. Baker and H. Davenport. The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$. *Quart. J. Math. Oxford Ser. (2)*, 20:129–137, 1969.
- [2] C. Bertók, L. Hajdu, I. Pink, and Z. Rábai. Linear combinations of prime powers in binary recurrence sequences. to appear in *International Journal of Number Theory*.
- [3] J. J. Bravo, C. A. Gómez, and F. Luca. Powers of two as sums of two k -Fibonacci numbers. available at arxiv.org/abs/1409.8514.
- [4] J. J. Bravo and F. Luca. On the Diophantine equation $F_n + F_m = 2^a$. to appear in *Questiones Mathematicae*.
- [5] J. J. Bravo and F. Luca. Powers of two as sums of two Lucas numbers. *J. Integer Seq.*, 17(8):Article 14.8.3, 12, 2014.
- [6] Y. Bugeaud and M. Laurent. Minoration effective de la distance p -adique entre puissances de nombres algébriques. *J. Number Theory*, 61(2):311–342, 1996.
- [7] B. M. M. de Weger. Solving exponential Diophantine equations using lattice basis reduction algorithms. *J. Number Theory*, 26(3):325–367, 1987.
- [8] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász. Factoring polynomials with rational coefficients. *Math. Ann.*, 261(4):515–534, 1982.
- [9] D. Marques. Powers of two as sum of two generalized Fibonacci numbers. available at [http://arxiv.org/abs/1409.2704](https://arxiv.org/abs/1409.2704).
- [10] E. M. Matveev. An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II. *Izv. Ross. Akad. Nauk Ser. Mat.*, 64(6):125–180, 2000.
- [11] M. Mignotte and N. Tzanakis. Arithmetical study of recurrence sequences. *Acta Arith.*, 57(4):357–364, 1991.
- [12] A. Pethő and B. M. M. de Weger. Products of prime powers in binary recurrence sequences. I. The hyperbolic case, with an application to the generalized Ramanujan-Nagell equation. *Math. Comp.*, 47(176):713–727, 1986.
- [13] A. Schinzel. On two theorems of Gelfond and some of their applications. *Acta Arith.*, 13:177–236, 1967/1968.
- [14] N. P. Smart. *The algorithmic resolution of Diophantine equations*, volume 41 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1998.
- [15] L. C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.
- [16] K. Yu. p -adic logarithmic forms and group varieties. II. *Acta Arith.*, 89(4):337–378, 1999.
- [17] K. Yu. p -adic logarithmic forms and a problem of Erdős. *Acta Math.*, 211(2):315–382, 2013.

I. PINK

INSTITUTE OF MATHEMATICS, UNIVERSITY OF DEBRECEN

H-4010 DEBRECEN, P.O. BOX 12, HUNGARY

AND

UNIVERSITY OF SALZBURG

HELLBRUNNERSTRASSE 34/I

A-5020 SALZBURG, AUSTRIA

E-mail address: pinki@science.unideb.hu; istvan.pink@sbg.ac.at

V. ZIEGLER
 UNIVERSITY OF SALZBURG
 HELLBRUNNERSTRASSE 34/I
 A-5020 SALZBURG, AUSTRIA
E-mail address: `volker.ziegler@sbg.ac.at`